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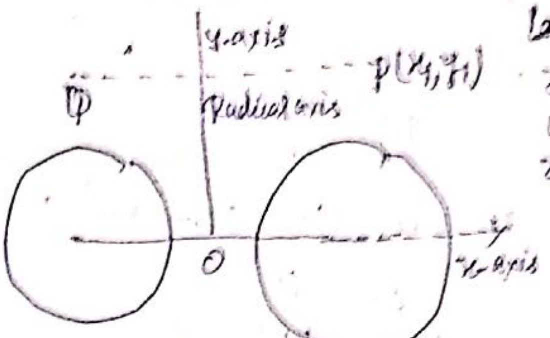
B.Sc-part-I (H) paper-II Date-12-04-2020

Analytical Geometry of two dimensions

Q.1: The polars of a point P with respect to two given circles meet in Q. Show that the radical axis of the circles bisects PQ.

Soln:



Let us take the line joining the centres of the circles as x-axis and the radical axis as y-axis, that is, $x=0$

Then, the equations of the circles can be written as

$$x^2+y^2+2g_1x+c=0 \quad \text{——— (I)}$$

and

$$x^2+y^2+2g_2x+c=0 \quad \text{——— (II)}$$

Let P be (x_1,y_1). Then, the polars of P with respect to the circles I and II are

$$xx_1+yy_1+g_1(x+x_1)+c=0 \quad \text{——— (III)}$$

and

$$xx_1+yy_1+g_2(x+x_1)+c=0 \quad \text{——— (IV)}$$

Subtracting (IV) from III, we get

$$(x+x_1)(g_1-g_2)=0$$

But $g_1 \neq g_2$, as the circles (I) and II are different

$$\therefore x=-x_1$$

The x-co-ordinate of the mid-point of PQ $= \dfrac{x_1+(-x_1)}{2}=0$,

which lies on the axis of y, that is, the radical axis.

Q.2. Prove that the common tangent to two circles of a co-axial system subtends a right angle at ~~that~~ either limiting point of the system.

Soln: let the two circles of a co-axial system be

$x^2+y^2+2g_1x+c=0$ and $x^2+y^2+2g_2x+c=0$

Let the common tangent touch these circles at L and M and cut y-axis at $P(0,y_1)$.

Then, $PL^2 = y_1^2+c$ and $PM^2 = y_1^2+c$

The limiting points, A and B are respectively

$(\sqrt{c},0)$ and $(-\sqrt{c},0)$

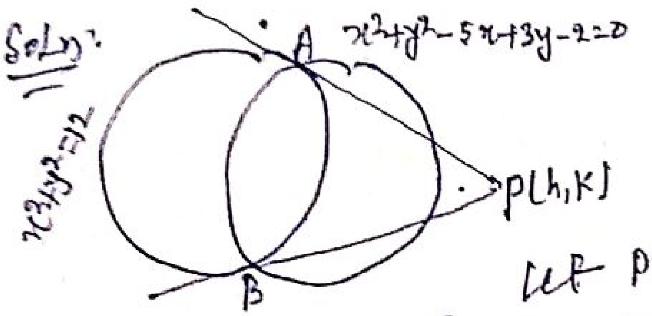
Now, $PA^2 = y_1^2+c$ and $PB^2 = y_1^2+c$

In $\triangle LAM$, we find that $PL=PM=PA$ and P is the mid-point of LM. $\therefore \angle LAM = 90^\circ$

Similarly, we can prove that $\angle LBM = 90^\circ$

Q.3. Tangents are drawn to the circle $x^2+y^2=12$ at the points where it is met by the circle $x^2+y^2-5x+3y-2=0$, find the point of intersection of these tangents.

Soln:



The equation to the common chord AB of the given circles is

$x^2+y^2-12-(x^2+y^2-5x+3y-2)=0$

$\Rightarrow 5x-3y-10=0$ —— I

Let $P(h,k)$ be the point of intersection of the tangents drawn to the circle $x^2+y^2=12$.

Then, AB is the chord of contact of tangents drawn from P.

Therefore, the equation of AB is

$xh+yk-12=0$ —— II

Comparing I and II, we get

$\frac{h}{5} = \frac{k}{-3} = \frac{-12}{-10} \Rightarrow h=6$ and $k=-\frac{18}{5}$

Hence, the required point is $\left(6,-\frac{18}{5}\right)$

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B.Sc. part-II (H) paper-III, Date-12-04-2020

Differential Calculus

Q.1: If $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$, prove that

$$(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$$

Soln: It is given that

$$y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$$

Differentiating the given relation,

$$\frac{1}{m} y^{\frac{1}{m}-1} y_1 - \frac{1}{m} y^{-\frac{1}{m}-1} y_1 = 2x$$

$$\Rightarrow \frac{1}{m} y_1 y^{-1} \left[y^{\frac{1}{m}} - y^{-\frac{1}{m}}\right] = 2x$$

$$\Rightarrow \frac{y_1}{my} \left[y^{\frac{1}{m}} - y^{-\frac{1}{m}}\right] = 2$$

Squaring both sides, we get

$$y_1^2 \left[y^{\frac{1}{m}} - y^{-\frac{1}{m}}\right]^2 = (2my)^2$$

$$\Rightarrow y_1^2 \left[\left(y^{\frac{1}{m}} + y^{-\frac{1}{m}}\right)^2 - 4 y^{\frac{1}{m}} y^{-\frac{1}{m}}\right] = 4m^2y^2$$

$$\Rightarrow y_1^2 [4x^2-4] - 4m^2y^2 \Rightarrow y_1^2 (x^2-1) = m^2y^2$$

Again, differentiating, we get

$$2y_1 y_2 (x^2-1) + y_1^2 (2x) = 2m^2 y y_1$$

$$\Rightarrow (x^2-1)y_2 + xy_1 - m^2y = 0$$

Differentiating n times by Leibnitz's Theorem

$$(x^2-1)y_{n+2} + {}^nC_1(2x)y_{n+1} + {}^nC_2(2)y_n + xy_{n+1} + {}^nC_1 y_n - m^2y_n = 0$$

$$\Rightarrow (x^2-1)y_{n+2} + xy_{n+1}[2\,{}^nC_1 + 1] + y_n[2\,{}^nC_2 + {}^nC_1 - m^2] = 0$$

$$\Rightarrow (x^2-1)y_{n+2} + xy_{n+1}(2n+1) + y_n(n^2-m^2) = 0.$$

Proved.

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Differential Calculus.

Q.2. If $y = e^{a\sin^{-1}x}$, prove that

(I) $(1-x^2)y_2 - xy_1 - a^2y = 0$ (II) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$

Soln: We have, $\because y = e^{a\sin^{-1}x}$ ——— (I)

Differentiating, we get

$y_1 = e^{a\sin^{-1}x} \times \frac{a}{\sqrt{1-x^2}}$ ——— II

$\Rightarrow \sqrt{1-x^2}\cdot y_1 = a e^{a\sin^{-1}x} = ay$ by I

$\therefore (1-x^2)y_1^2 = a^2y^2$

Again, differentiating, we get

$(1-x^2)2y_1y_2 + (-2x)y_1^2 = a^2 2yy_1$

$\Rightarrow (1-x^2)y_2 - xy_1 - a^2y = 0$ proved.

Differentiating n times by Leibnitz's Theorem.

$(1-x^2)y_{n+2} + {}^nC_1(-2x)y_{n+1} + {}^nC_2(-2)y_n - xy_{n+1} - {}^nC_1 y_n \cdot 1 - a^2y_n = 0$

$\Rightarrow (1-x^2)y_{n+2} - xy_{n+1}[2\,{}^nC_1 + 1] - y_n[2\,{}^nC_2 + {}^nC_1 + a^2] = 0$

$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$

proved

Q.3. If $y = a\cos(\log x) + b\sin(\log x)$, prove that

(I) $x^2y_2 + xy_1 + y = 0$. (II) $x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$.

Soln:- We have, $y = a\cos(\log x) + b\sin(\log x)$

$\therefore y_1 = -a\sin(\log x)\cdot\frac{1}{x} + b\cos(\log x)\cdot\frac{1}{x}$

i.e. $xy_1 = -a\sin(\log x) + b\cos(\log x)$

$\therefore xy_2 + y_1 = -a\cos(\log x)\frac{1}{x} - b\sin(\log x)\frac{1}{x}$

i.e. $x^2y_2 + xy_1 = -[a\cos(\log x) + b\sin(\log x)] = -y$

$\therefore x^2y_2 + xy_1 + y = 0$ ——— I

Differentiating (I) n times with respect to x by Leibnitz's theorem, we get

$y_{n+2}x^2 + \frac{n}{1}y_{n+1}(2x) + \frac{n(n-1)}{1\cdot 2}y_n \cdot 2 + y_{n+1}x + \frac{n}{1}y_n \cdot 1 + y_n = 0$

$\Rightarrow x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$

proved